國科大杭州高等研究院
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基础物理与数学科学学院
School of Fundamental Physics and Mathematical Sciences

作业 1

课　　程　广义相对论

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Exercise 1:

已知：$A^\mu(x)$ 和 $B^\mu(x)$ 是两个光滑的逆变矢量场。

需证：其对易子 $[A,B]^\mu \equiv A^\nu\partial_\nu B^\mu - B^\nu\partial_\nu A^\mu$ 满足线性性和莱布尼兹律，且在坐标变换下符合逆变矢量的变换规律，从而证明对易子也是一个逆变矢量场。

证明如下：

将对易子 $[A,B]^\mu$ 作用于任意函数 f,g 上, 对流形上的每一点都有

线性性：

$$\begin{aligned}[A,B](\alpha f+\beta g) &= A[B(\alpha f+\beta g)] - B[A(\alpha f+\beta g)] \\ &= \alpha[A,B](f)+\beta[A,B](g)\end{aligned} \tag{1}$$

莱布尼兹律：

$$\begin{aligned}[A,B](fg) &= A^\nu\partial_\nu\left[B^\mu\partial_\mu(fg)\right] - B^\nu\partial_\nu\left[A^\mu\partial_\mu(fg)\right] \\ &= fA^\nu\partial_\nu(B^\mu\partial_\mu g) + gA^\nu\partial_\nu(B^\mu\partial_\mu f) - fB^\nu\partial_\nu(A^\mu\partial_\mu g) - gB^\nu\partial_\nu(A^\mu\partial_\mu f) \\ &= f[A,B](g)+g[A,B](f)\end{aligned} \tag{2}$$

逆变矢量场在坐标变换下的规律是

$$A^\mu \longrightarrow A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}A^\nu \tag{3}$$

而对易子在坐标变换下变为

$$\begin{aligned}[A,B]'^\mu &= A'^\nu\partial'_\nu B'^\mu - B'^\nu\partial'_\nu A'^\mu \\ &= \frac{\partial x'^\nu}{\partial x^\sigma}A^\sigma\frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial}{\partial x^\rho}(\frac{\partial x'^\mu}{\partial x^\alpha}B^\alpha) - \frac{\partial x'^\nu}{\partial x^\sigma}B^\sigma\frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial}{\partial x^\rho}(\frac{\partial x'^\mu}{\partial x^\alpha}A^\alpha) \\ &= \delta^\rho{}_\sigma[\frac{\partial x'^\mu}{\partial x^\alpha}(A^\sigma\partial_\rho B^\alpha - B^\sigma\partial_\rho A^\alpha) + \frac{\partial^2 x'^\mu}{\partial x^\rho\partial x^\alpha}(A^\sigma B^\alpha - B^\sigma A^\alpha)] \\ &= \frac{\partial x'^\mu}{\partial x^\alpha}(A^\rho\partial_\rho B^\alpha - B^\rho\partial_\rho A^\alpha) \\ &= \frac{\partial x'^\mu}{\partial x^\alpha}[A,B]^\alpha\end{aligned} \tag{4}$$

故逆变矢量场的对易子也是一个逆变矢量场。

Exercise 2:

(a)

已知：对任意矢量场 V^μ 可令 $S^{\mu\nu}=V^\mu V^\nu$。

需证：$S^{\mu\nu}$ 关于 $\mu\nu$ 指标对称。

证明如下：

$$S^{(\mu\nu)} = \frac{1}{2}\left(S^{\mu\nu}+S^{\nu\mu}\right) = \frac{1}{2}\left(V^\mu V^\nu + V^\nu V^\mu\right) = \frac{1}{2}\times 2\times V^\mu V^\nu = V^\mu V^\nu = S^{\mu\nu} \tag{5}$$

(b)

需证：$T_{\mu\nu}$ 是反对称张量是对任意矢量 V^μ 都有 $T_{\mu\nu}V^\mu V^\nu$ 的充分必要条件。

证明如下：

充分性——由于 $T_{\mu\nu}$ 是反对称的，$V^\mu V^\nu$ 是对称的，所以对任意矢量 V^μ

$$T_{\mu\nu}V^\mu V^\nu = T_{[\mu\nu]}V^{(\mu}V^{\nu)} = 0 \tag{6}$$

必要性——根据对称指标的传递性

$$T_{\mu\nu}V^\mu V^\nu = T_\mu V^{(\mu}V^{\nu)} = T_{(\mu\nu)}V^\mu V^\nu = 0 \tag{7}$$

对任意矢量 V^μ 都成立，所以 $T_{(\mu\nu)} = 0$, 即 $T_{\mu\nu} = T_{[\mu\nu]}$.

Exercise 3:

已知：类空曲线 $C(t)$ 在某坐标下的参数式为 $x^\mu(t)$，线上两点 $p = C(t_1), q = C(t_2)$ 之间的线长为

$$l = \int_{t_1}^{t_2} \sqrt{g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}dt$$

求出：极值曲线所满足的方程。

解答如下：

$$\begin{aligned}
\delta l = 0 &= \int_{t_1}^{t_2} \delta\left[g_{\mu\nu}dx^\mu dx^\nu\right]^{\frac{1}{2}} = \frac{1}{2}\int_{t_1}^{t_2}\left(g_{\mu\nu}dx^\mu dx^\nu\right)^{-\frac{1}{2}}\delta\left(g_{\mu\nu}dx^\mu dx^\nu\right) \\
&= \frac{1}{2}\int\frac{1}{dl}\delta\left[g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}(dt)^2\right] = \frac{1}{2}\int dt\frac{dt}{dl}\delta\left(g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right) \\
&= \frac{1}{2}\int dt\frac{dt}{dl}\left[2g_{\mu\nu}\frac{dx^\mu}{dt}\frac{d\delta x^\nu}{dt} + \delta g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right] \\
&\simeq \frac{1}{2}\int dt\frac{dt}{dl}\left\{2\frac{d}{dt}\left[g_{\mu\nu}\frac{dx^\mu}{dt}\frac{d\delta x^\nu}{dt}\right] - 2\frac{d}{dt}\left(g_{\mu\nu}\frac{dx^\mu}{dt}\right)\delta x^\nu + \frac{\partial g_{\mu\nu}}{\partial x^\rho}\delta x^\rho\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right\} \\
&= \frac{1}{2}\int dt\frac{dt}{dl}\left[-2\left(g_{\mu\nu}\frac{d^2x^\mu}{dt^2} + \frac{dg_{\mu\nu}}{dt}\frac{dx^\mu}{dt}\right)\delta x^\nu + \frac{\partial g_{\mu\nu}}{\partial x^\rho}\delta x^\rho\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right] \\
&= \frac{1}{2}\int dt\frac{dt}{dl}\left[-2\left(g_{\mu\nu}\frac{d^2x^\mu}{dt^2} + \frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\rho}{dt}\frac{dx^\mu}{dt}\right)\delta x^\nu + \frac{\partial g_{\mu\nu}}{\partial x^\rho}\delta x^\rho\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right] \\
&= -\int dt\frac{dt}{dl}\left[g_{\mu\rho}\frac{d^2x^\mu}{dt^2} + \frac{\partial g_{\mu\rho}}{\partial x^\nu}\frac{dx^\nu}{dt}\frac{dx^\mu}{dt} - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right]\delta x^\rho \\
&= -\int dt\frac{dt}{dl}\left[g_{\mu\rho}\frac{d^2x^\mu}{dt^2} + \frac{1}{2}\left(\frac{\partial g_{\nu\rho}}{\partial x^\mu} + \frac{\partial g_{\mu\rho}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right]\delta x^\rho
\end{aligned} \tag{8}$$

同乘 $g^{\rho\sigma}$，由 δx^ρ 的任意性，得到粒子的运动方程

$$\frac{d^2x^\sigma}{dt^2} + \frac{1}{2}g^{\rho\sigma}\left(\frac{\partial g_{\nu\rho}}{\partial x^\mu} + \frac{\partial g_{\mu\rho}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 0 \tag{9}$$

即极值曲线所满足的方程为测地线方程。

Exercise 4:

已知：二维欧氏空间度规为 $ds^2 = dx^2 + dy^2$，有标量场 $f(x, y)$.

计算：$d * df$，其中 d 为外微分，$*$ 为 Hodge star 对偶。

解答如下：

1 形式

$$df = \nabla_\mu f = \partial_\mu f \ dx^\mu = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \tag{10}$$

作用 Hodge star 对偶后还是一个 1 形式

$$(*df)_a = (df)^b \varepsilon_{ba} \tag{11}$$

代入二维欧氏空间的体元 $\varepsilon_{12} = -\varepsilon_{21} = 1, \varepsilon_{11} = \varepsilon_{22} = 0$

$$*df = -\frac{\partial f}{\partial y} dx + \frac{\partial f}{\partial x} dy \tag{12}$$

根据梁书 (5-1-13) 对 1 形式求外微分 $d(Xdx + Ydy) = (\frac{\partial Y}{\partial X} - \frac{\partial X}{\partial y}) dx \wedge dy$，可以得到

$$d * df = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) dx \wedge dy \tag{13}$$

即在二维欧氏空间中对标量场 f 求拉普拉斯算子的作用。

Exercise 5:

需证：对于 1 形式场 ω_a 李导数和外微分的作用可交换，即 $\mathcal{L}_v \circ d = d \circ \mathcal{L}_v$.

证明如下：

1 形式场 $\omega_a = \omega_\mu (dx^\mu)_a$ 是对偶矢量场，其与任意一个矢量场 $V^a = V^\mu (\frac{\partial}{\partial x^\mu})^a$ 的缩并是一个标量，求外微分：

$$d[\omega(V)] = d(\omega_\mu V^\mu) = \partial_\nu (\omega_\mu V^\mu) dx^\nu = (V^\mu \partial_\nu \omega_\mu + \omega_\mu \partial_\nu V^\mu) dx^\nu \tag{14}$$

$(d\omega)_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu$ 作为一个 2 形式场，与矢量场 V 做内积 $(\iota_V \omega)_\nu = V^\mu \omega_{\mu\nu}$ 得到一个 1 形式场

$$(d\omega)(V) = (V^\mu \partial_\mu \omega_\nu - V^\mu \partial_\nu \omega_\mu) dx^\nu \tag{15}$$

另有 1 形式场的李导数即是对偶矢量场的李导数

$$\mathcal{L}_V \omega_\mu = V^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu V^\nu = [(d\omega)(V) + (d\omega)(V)]_\mu \tag{16}$$

即嘉当公式在 1 形式场的层面上成立。进一步有

$$d \circ \mathcal{L}_V(\omega) = d^2[\omega(V)] + d[(d\omega)(V)] \tag{17}$$

$$\mathcal{L}_V \circ d(\omega) = d[(d\omega)(V)] + (d^2\omega)(V) \tag{18}$$

利用外导数的幂零性 $d^2 = 0$，最终证得对 1 形式场 ω，有

$$d \circ \mathcal{L}_V(\omega) = \mathcal{L}_V \circ d(\omega) \tag{19}$$

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作业 2

课　程　广义相对论

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Exercise 1:

已知：联络在不同坐标系的变换关系

$$\tilde{\Gamma}^{\tau}{}_{lk}(\tilde{x}) = \frac{\partial \tilde{x}^{\tau}}{\partial x^{\rho}} \frac{\partial x^{\mu}}{\partial \tilde{x}^{l}} \frac{\partial x^{\sigma}}{\partial \tilde{x}^{k}} \Gamma^{\rho}{}_{\mu\sigma}(x) + \frac{\partial \tilde{x}^{\tau}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial \tilde{x}^{k} \partial \tilde{x}^{l}} \tag{1}$$

需证：对任意协变矢量场 B_μ 的协变导数

$$\nabla_\lambda B_\mu \equiv \partial_\lambda B_\mu - \Gamma^{\sigma}{}_{\mu\lambda} B_\sigma \tag{2}$$

满足张量变换律，从而是一个 (0,2) 型张量。

证明如下：

协变矢量场在不同坐标系的变换规律有

$$\tilde{B}_\mu = \frac{\partial x^{\rho}}{\partial \tilde{x}^{\mu}} B_\rho \tag{3}$$

其普通导数

$$\frac{\partial}{\partial \tilde{x}^{\lambda}} \tilde{B}_\mu = \frac{\partial x^{\sigma}}{\partial \tilde{x}^{\lambda}} \frac{\partial}{\partial x^{\sigma}} \left(\frac{\partial x^{\rho}}{\partial \tilde{x}^{\mu}} B_\rho \right) = \frac{\partial x^{\sigma}}{\partial \tilde{x}^{\lambda}} \frac{\partial x^{\rho}}{\partial \tilde{x}^{\mu}} \frac{\partial B_\rho}{\partial x^{\sigma}} + \frac{\partial x^{\sigma}}{\partial \tilde{x}^{\lambda}} \frac{\partial^2 x^{\rho}}{\partial x^{\sigma} \partial \tilde{x}^{\mu}} B_\rho \tag{4}$$

第二项的变换规律为

$$\tilde{\Gamma}^{\sigma}{}_{\mu\lambda}(\tilde{x}) \tilde{B}_\sigma(\tilde{x}) = \left(\frac{\partial \tilde{x}^{\sigma}}{\partial x^{\beta}} \frac{\partial x^{\rho}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\nu}}{\partial \tilde{x}^{\lambda}} \Gamma^{\beta}{}_{\rho\nu} + \frac{\partial \tilde{x}^{\sigma}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial \tilde{x}^{\mu} \tilde{x}^{\lambda}} \right) \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\sigma}} B_\alpha \tag{5}$$

由此协变矢量场的协变导数在坐标变换下的变换规律为

$$\begin{aligned}
\tilde{\nabla}_\lambda \tilde{B}_\mu &= \frac{\partial x^{\sigma}}{\partial \tilde{x}^{\lambda}} \frac{\partial x^{\rho}}{\partial \tilde{x}^{\mu}} \partial_\sigma B_\rho - \delta^{\alpha}{}_{\beta} \frac{\partial x^{\rho}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\sigma}}{\partial \tilde{x}^{\lambda}} \Gamma^{\beta}{}_{\rho\sigma} B_\alpha + \frac{\partial x^{\sigma}}{\partial \tilde{x}^{\lambda}} \frac{\partial^2 x^{\rho}}{\partial x^{\sigma} \partial \tilde{x}^{\mu}} B_\rho - \delta^{\alpha}{}_{\rho} \frac{\partial^2 x^{\rho}}{\partial \tilde{x}^{\mu} \partial \tilde{x}^{\lambda}} B_\alpha \\
&= \frac{\partial x^{\sigma}}{\partial \tilde{x}^{\lambda}} \frac{\partial x^{\rho}}{\partial \tilde{x}^{\mu}} \left(\partial_\sigma B_\rho - \Gamma^{\beta}{}_{\rho\sigma} B_\beta \right) \\
&= \frac{\partial x^{\sigma}}{\partial \tilde{x}^{\lambda}} \frac{\partial x^{\rho}}{\partial \tilde{x}^{\mu}} \nabla_\sigma B_\rho
\end{aligned} \tag{6}$$

故 $\nabla_\lambda B_\mu$ 是一个 (0,2) 型张量。

Exercise 2:

(a)

已知：测地线 $\gamma(t)$ 上的切矢为 t^a

需证：测地线重参数化为 $\tilde{\gamma(t)}$ 的切矢 $\tilde{t}^a$ 满足

$$\tilde{t}^b \nabla_b \tilde{t}^a = \alpha \tilde{t}^a \tag{7}$$

其中 α 为 $\gamma(t)$ 上的某函数。

证明如下：

测地线切矢满足自平行条件

$$t^b \nabla_b t^a = 0 \tag{8}$$

重参数化后的切矢与原参数表示的切矢关系为

$$\tilde{t}^a \equiv \left(\frac{\partial}{\partial \tilde{t}}\right)^a = \frac{dt}{d\tilde{t}}\left(\frac{\partial}{\partial t}\right)^a = \frac{dt}{d\tilde{t}} t^a \tag{9}$$

即 $t^a = \tilde{t}^a \frac{d\tilde{t}}{dt}$。代入自平行条件：

$$\begin{aligned}
0 &= \left(\tilde{t}^b \frac{d\tilde{t}}{dt}\right) \nabla_b \left(\tilde{t}^a \frac{d\tilde{t}}{dt}\right) \\
&= \left(\frac{d\tilde{t}}{dt}\right)^2 \tilde{t}^b \nabla_b \tilde{t}^a + \tilde{t}^a \left(\frac{d\tilde{t}}{dt}\right) \tilde{t}^b \nabla_b \frac{d\tilde{t}}{dt} \\
&= \left(\frac{d\tilde{t}}{dt}\right)^2 \tilde{t}^b \nabla_b \tilde{t}^a + \tilde{t}^a \left(\frac{d\tilde{t}}{dt}\right) \frac{d}{d\tilde{t}} \left(\frac{d\tilde{t}}{dt}\right) \\
&= \left(\frac{d\tilde{t}}{dt}\right)^2 \tilde{t}^b \nabla_b \tilde{t}^a + \tilde{t}^a \frac{d^2\tilde{t}}{dt^2}
\end{aligned} \tag{10}$$

即

$$\tilde{t}^a \nabla_a \tilde{t}^b = -\left(\frac{dt}{d\tilde{t}}\right)^2 \left(\frac{d^2\tilde{t}}{dt^2}\right) \tilde{t}^b \tag{11}$$

其中 $\alpha \equiv -\left(\frac{dt}{d\tilde{t}}\right)^2 \left(\frac{d^2\tilde{t}}{dt^2}\right)$。证毕。

(b)

需证：非类光测地线的线长参数必为仿射参数。

证明如下：先证以仿射参数为参数的测地线切矢长度沿测地线为常数，即

$$t^a \nabla_a \left(g_{bc} t^b t^c\right) = t^a t^b t^c \nabla_a g_{bc} + g_{bc} t^c t^a \nabla_a t^b + g_{bc} t^b t^a \nabla_a t^c \tag{12}$$

根据协变导数与度规的适配性 $\nabla_a g_{bc} = 0$ 和仿射参数下的测地线切矢所满足的自平行条件 $t^a \nabla_a t^b = 0$，上式为零，说明切矢长度 $|\, T \,| \equiv g_{bc} t^b t^c$ 沿测地线不变。线长 l 与仿射参数 t 之间的关系由梁书 (2-5-2) 给出

$$dl = |\, T \,|\, dt \tag{13}$$

t 为仿射参数。根据梁书定理 3-3-3，线长 l 与 t 线性关联，故线长参数也是仿射参数。

Exercise 3:

已知：背景时空是无挠的，$T^{\mu}{}_{\nu\lambda} \equiv \Gamma^{\mu}{}_{\nu\lambda} - \Gamma^{\mu}{}_{\lambda\nu} = 0$

(a)

需证：$R_{\mu\nu\kappa\lambda} = -R_{\nu\mu\kappa\lambda}$

证明如下：

克里斯托菲符号表示为

$$\Gamma^{\rho}{}_{\mu\lambda} = \frac{1}{2} g^{\rho\tau} \left(g_{\mu\tau,\lambda} + g_{\lambda\tau,\mu} - g_{\mu\lambda,\tau} \right) \tag{14}$$

黎曼曲率张量定义为

$$R^{\nu}{}_{\mu\kappa\lambda} \equiv \partial_k \Gamma^{\nu}{}_{\mu\lambda} - \partial_\lambda \Gamma^{\nu}{}_{\mu\kappa} + \Gamma^{\nu}{}_{\sigma\kappa}\Gamma^{\sigma}{}_{\mu\lambda} - \Gamma^{\nu}{}_{\sigma\lambda}\Gamma^{\sigma}{}_{\mu\kappa} \tag{15}$$

因而有

$$\begin{aligned} R_{\nu\mu\kappa\lambda} = g_{\nu\rho} R^{\rho}{}_{\mu\kappa\lambda} &= g_{\nu\rho}\partial_\kappa \Gamma^{\rho}{}_{\mu\lambda} - g_{\nu\rho}\partial_\lambda \Gamma^{\rho}{}_{\mu\kappa} + g_{\nu\rho}\Gamma^{\nu}{}_{\sigma\kappa}\Gamma^{\sigma}{}_{\mu\lambda} - g_{\nu\rho}\Gamma^{\nu}{}_{\sigma\lambda}\Gamma^{\sigma}{}_{\mu\kappa} \\ &\xlongequal{\text{在局域惯性系下}} g_{\nu\rho}\partial_\kappa \Gamma^{\rho}{}_{\mu\lambda} - g_{\nu\rho}\partial_\lambda \Gamma^{\rho}{}_{\mu\kappa} \\ &= g_{\nu\rho}[\frac{1}{2} g^{\rho\tau}{}_{,\kappa} \left(g_{\tau\mu,\lambda} + g_{\tau\lambda,\mu} - g_{\mu\lambda,\tau} \right) + \frac{1}{2} g^{\rho\tau} \left(g_{\mu\tau,\lambda\kappa} + g_{\lambda\tau,\mu\kappa} - g_{\mu\lambda,\tau\kappa} \right) \\ &\quad - \frac{1}{2} g^{\rho\tau}{}_{,\lambda} \left(g_{\mu\tau,\kappa} + g_{\kappa\tau,\mu} - g_{\mu\kappa,\tau} \right) - \frac{1}{2} g^{\rho\tau} \left(g_{\mu\tau,\kappa\lambda} + g_{\kappa\tau,\mu\lambda} - g_{\mu\kappa,\tau\lambda} \right)] \\ &= -g_{\nu\rho,\kappa}\Gamma^{\rho}{}_{\mu\lambda} + g_{\upsilon\rho,\lambda}\Gamma^{\rho}{}_{\mu\kappa} + \frac{1}{2} \left(g_{\nu\lambda,\mu\kappa} + g_{\mu\kappa,\nu\lambda} - g_{\mu\lambda,\nu\kappa} - g_{\nu\kappa,\mu\lambda} \right) \\ &\xlongequal{\text{在局域惯性系下}} \frac{1}{2} \left(g_{\nu\lambda,\mu\kappa} + g_{\mu\kappa,\nu\lambda} - g_{\mu\lambda,\nu\kappa} - g_{\nu\kappa,\mu\lambda} \right) \end{aligned} \tag{16}$$

交换 $\mu\nu$ 指标，得到

$$R_{\mu\nu\kappa\lambda} \xlongequal{\text{在局域惯性系下}} \frac{1}{2} \left(g_{\mu\lambda,\nu\kappa} + g_{\nu\kappa,\mu\lambda} - g_{\nu\lambda,\mu\kappa} - g_{\mu\kappa,\nu\lambda} \right) \tag{17}$$

可知在局域惯性系下 $R_{\nu\mu\kappa\lambda} = -R_{\mu\nu\kappa\lambda}$。张量等式在一个坐标系下成立，就在所有坐标系下成立，所以

$$R_{\mu\nu\kappa\lambda} = -R_{\nu\mu\kappa\lambda} \tag{18}$$

对所有坐标系都成立。

(b)

需证：$R^{\nu}{}_{[\mu\kappa\lambda]} = 0$

证明如下：

$$R^{\nu}{}_{[\mu k\lambda]} = \frac{1}{6} \left(R^{\nu}{}_{\mu\kappa\lambda} + R^{\nu}{}_{\lambda\mu\kappa} + R^{\nu}{}_{\kappa\lambda\mu} - R^{\nu}{}_{\mu\lambda\kappa} - R^{\nu}{}_{\kappa\mu\lambda} - R^{\nu}{}_{\lambda\kappa\mu} \right) \tag{19}$$

利用

$$R^{\nu}{}_{\mu\kappa\lambda} + R^{\nu}{}_{\lambda\mu\kappa} + R^{\nu}{}_{\kappa\lambda\mu} \xlongequal{\text{在局域惯性系下}} \Gamma^{\nu}{}_{\mu\lambda,\kappa} - \Gamma^{\nu}{}_{\mu\kappa,\lambda} + \Gamma^{\nu}{}_{\kappa\mu,\lambda} - \Gamma^{\nu}{}_{\kappa\lambda,\mu} + \Gamma^{\nu}{}_{\lambda\kappa,\mu} - \Gamma^{\nu}{}_{\lambda\mu,\kappa} = 0 \tag{20}$$

交换指标后可以得证。

(c)

需证：$\nabla_\mu \nabla_\kappa \xi_\lambda = R^{\nu}{}_{\mu\kappa\lambda}\xi_\nu$，其中 ξ_ν 为 Killing 矢量场。

证明如下：

对任意的协变矢量场，黎曼曲率张量的作用定义为

$$
\left(\nabla_\mu \nabla_k - \nabla_k \nabla_\mu\right) \xi_\lambda = -R^\nu{}_{\lambda\mu\kappa} \xi_\nu \tag{21}
$$

此处 ξ_ν 是一个 Killing 矢量场，满足 Killing 方程

$$
\nabla_\nu \xi_\lambda = -\nabla_\lambda \xi_\nu \tag{22}
$$

代入定义式后对 μ, κ, μ 三个指标进行轮换：

$$
\begin{aligned}
\nabla_\mu \nabla_\kappa \xi_\lambda + \nabla_\kappa \nabla_\lambda \xi_\mu = -R^\nu{}_{\lambda\mu\kappa} \xi_\nu \\
\nabla_\lambda \nabla_\mu \xi_\kappa + \nabla_\mu \nabla_\kappa \xi_\lambda = -R^\nu{}_{\kappa\lambda\mu} \xi_\nu \\
\nabla_\kappa \nabla_\lambda \xi_\mu + \nabla_\lambda \nabla_\mu \xi_\kappa = -R^\nu{}_{\mu\kappa\lambda} \xi_\nu
\end{aligned} \tag{23}
$$

第一式加第二式减第三式，再利用 Ricci 恒等式 $R^\nu_{\mu\kappa\lambda} + R^\nu{}_{\lambda\mu\kappa} + R^\nu{}_{\kappa\lambda\mu} = 0$，可以得到

$$
2\nabla_\mu \nabla_k \xi_\lambda = \left(-R^\nu{}_{\lambda\mu\kappa} - R^\nu{}_{\kappa\lambda\mu} + R^\nu{}_{\mu\kappa\lambda}\right) \xi_\nu = 2R^\nu{}_{\mu\kappa\lambda} \xi_\nu \tag{24}
$$

即

$$
\nabla_\mu \nabla_k \xi_\lambda = R^\nu{}_{\mu\kappa\lambda} \xi_\nu \tag{25}
$$

得证。

Exercise 4:

已知：二维球面度规为 $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$。

计算：这个度规的克氏符 $\Gamma^\rho{}_{\mu\nu}$、黎曼曲率张量 $R^\mu{}_{\nu\lambda\rho}$、里奇张量 $R_{\mu\nu}$ 和曲率标量 R。

计算如下：度规的非零分量有

$$
g_{\theta\theta} = 1,\ g_{\phi\phi} = \sin^2\theta \tag{26}
$$

给出克氏符

$$
\Gamma^\nu{}_{\mu\lambda} = \frac{1}{2} g^{\nu\kappa} \left(g_{\mu\kappa,\lambda} + g_{\lambda\kappa,\mu} - g_{\mu\lambda,\kappa}\right) \tag{27}
$$

非零分量有

$$
\Gamma^\phi{}_{\phi\theta} = \Gamma^\phi{}_{\theta\phi} = \cot\theta,\ \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta \tag{28}
$$

给出黎曼曲率张量

$$
R^\nu{}_{\mu\kappa\lambda} = \partial_\kappa \Gamma^\nu{}_{\mu\lambda} - \partial_\lambda \Gamma^\nu{}_{\mu\kappa} + \Gamma^\nu{}_{\sigma\kappa} \Gamma^\sigma{}_{\mu\lambda} - \Gamma^\nu_{\sigma\lambda} \Gamma^\sigma{}_{\mu\kappa} \tag{29}
$$

非零分量有

$$
R^\phi{}_{\theta\theta\phi} = -R^\phi{}_{\theta\phi\theta} = -1,\ R^\theta{}_{\phi\theta\phi} = -R^\theta{}_{\phi\phi\theta} = \sin^2\theta \tag{30}
$$

给出里奇张量

$$R_{\mu\lambda} = g_{\nu\kappa} R^{\nu}{}_{\mu\kappa\lambda} \tag{31}$$

非零分量有

$$R_{\phi\phi} = \sin^2\theta,\ R_{\theta\theta} = 1 \tag{32}$$

给出曲率标量

$$R = g^{\mu\lambda} R_{\mu\lambda} = 1 + \frac{1}{\sin^2\theta}\sin^2\theta = 2 \tag{33}$$

Exercise 5:

已知：背景时空是二维闵氏时空 $ds^2 = -dt^2 + dx^2$

求出：所有独立的 Killing 矢量场。

解答如下：

在 $\{t,x\}$ 坐标系下度规分量与坐标 t,x 都无关，所以 Killing 矢量场必有

$$\xi_1{}^a = \left(\frac{\partial}{\partial t}\right)^a \tag{34}$$

$$\xi_2{}^a = \left(\frac{\partial}{\partial x}\right)^a \tag{35}$$

求解 Killing 方程

$$\begin{cases} \partial_t \xi_x + \partial_x \xi_t = 0 \\ \partial_x \xi_x = 0 \\ \partial_t \xi_t = 0. \end{cases}$$

解出 $\xi_t = -bx + a_t, \xi_x = bt + a_x$，给出 $\xi_a = -x\left(\frac{\partial}{\partial t}\right)_a + t\left(\frac{\partial}{\partial x}\right)_a$，所以第三个 Killing 矢量场为

$$\xi_3{}^a = x\left(\frac{\partial}{\partial t}\right)^a + t\left(\frac{\partial}{\partial x}\right)^a \tag{36}$$

或取坐标系 $x = \psi\cosh\eta, t = \psi\sinh\eta$ 改写线元为 $ds^2 = d\psi^2 - \psi^2 d\eta^2$，度规与 η 无关可以得到同样的 $\xi_3{}^a$。

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作业 4

课　程	广义相对论
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学　号	2024280012O7031
日　期	2024.12.1

习题 1

对于施瓦西黑洞：

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

其中

$$f(r) = 1 - \frac{2M}{r}$$

计算其总质量，并验证对任意给定的视界外的体积 $V(r = const)$，得到的结果相同。
解：

施瓦西时空是渐近平直时空，其总质量需要利用 Komar 积分：

$$\oint_{\partial V} dS_{\mu\nu}\nabla^\mu\xi^\nu = \oint_{\partial V} dS_{\mu\nu}g^{\mu\sigma}\nabla_\sigma\xi^\nu = \oint_{\partial V} dS_{\mu\nu}g^{\mu\sigma}(\partial_\sigma\xi^\nu + \Gamma^\nu{}_{\sigma\rho}\xi^\rho) \tag{1}$$

其中 Killing 矢量场 $\xi^\nu = \left(\frac{\partial}{\partial t}\right)^\nu$，故 $\partial_\sigma\xi^\nu = 0$。下标含 0 的非零克氏符为

$$\Gamma^t{}_{tr} = \Gamma^t{}_{rt} = \frac{1}{2f(r)}f'(r), \Gamma^r{}_{tt} = \frac{1}{2}f(r)f'(r) \tag{2}$$

由此积分化简为

$$\begin{aligned}\oint_{\partial V} dS_{\mu\nu}g^{\mu\sigma}\Gamma^\nu{}_{\sigma\rho}\xi^\rho &= \oint_{\partial V}\left(dS_{rt}g^{rr}\Gamma^t{}_{rt} + dS_{tr}g^{tt}\Gamma^r{}_{tt}\right)\\ &= \oint_{\partial V}\left(dS_{rt}f(r)\frac{1}{2f(r)}f'(r) + dS_{tr}(-f^{-1}(r))\frac{1}{2}f(r)f'(r)\right)\\ &= \frac{1}{2}\oint_{\partial V}\left(dS_{rt}f'(r) - dS_{tr}f'(r)\right)\end{aligned} \tag{3}$$

面元的指标具有反对称性 $dS_{tr} = -dS_{rt}$，则有

$$\oint_{\partial V} dS_{\mu\nu}\nabla^\mu\xi^\nu = f'(r)\oint_{\partial V} S_{10} = -\frac{2M}{r^2}\int_0^{2\pi} d\phi\int_0^\pi r^2\sin\theta d\theta = -8\pi M \tag{4}$$

所以施瓦西黑洞的质量就是 M

$$M_K = -\frac{1}{8\pi}\oint_{\partial V} dS_{\mu\nu}\nabla^\mu\xi^\nu = M \tag{5}$$

在计算过程中并未假定时空 V 的范围，所以与对任意给定的视界外的体积均成立。

习题 2

对于 AdS 时空，证明未来类时测地线在有限固有时内无法到达边界 $\mathcal{J}$。（注：考虑静态坐标系中沿径向运动的粒子）
解：

在 AdS 时空的静态坐标系下，沿径向运动的粒子 $d\Omega = 0$ 感受到时空线元为：

$$ds^2 = -(1 + l^{-2}r^2)dt^2 + (1 + l^{-2}r^2)^{-1}dr^2 \tag{6}$$

其中 l 为 AdS 半径。对于类时测地线，切矢 U^μ 满足

$$U^\mu U_\mu = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1 \tag{7}$$

即

$$-(1+l^{-2}r^2)\left(\frac{dt}{d\tau}\right)^2 + (1+l^{-2}r^2)^{-1}\left(\frac{dr}{d\tau}\right)^2 = -1 \tag{8}$$

根据梁书定理 4-3-3，有守恒量

$$E = -g_{\mu\nu}\left(\frac{\partial}{\partial t}\right)^\mu\left(\frac{\partial}{\partial \tau}\right)^\nu = -\left[-(1+l^{-2}r^2)\right]\left(\frac{dt}{d\tau}\right) = (1+l^{-2}r^2)\frac{dt}{d\tau} \tag{9}$$

代入上述方程有

$$\left(\frac{dr}{d\tau}\right)^2 + l^{-2}r^2 = E^2 - 1 \tag{10}$$

解得

$$r(\tau) = \frac{l\sqrt{E^2-1}|\tan(\tau/l)|}{\sqrt{1+[\tan(\tau/l)]^2}} = l\sqrt{E^2-1}|\sin(\tau/l)| \tag{11}$$

可以发现固有时 τ 在趋于无穷的时候径向坐标 r 是有界的，即在有限固有时内未来类时测地线无法到达边界。

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作业 3

课　　程　广义相对论

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习题 1

在闵氏时空附近的物质场对度规作线性扰动

$$g_{\mu\nu}(x)=\eta_{\mu\nu}+h_{\mu\nu}(x)$$

在坐标变换 $x^{\mu}\longrightarrow x'^{\mu}=x^{\mu}+\epsilon^{\mu}(x)$ 下，其中 $|\,\epsilon^{\mu}\,|\ll 1$ 且其高阶导数也远小于 1，考虑弱场近似条件 $|\,h_{\mu\nu}\,|\ll 1$ 且其高阶导数也远小于 1。

(a) 求出 $h'_{\mu\nu}(x')$ 和 $h_{\mu\nu}(x)$ 之间的变换关系。

(b) 写出黎曼张量 $R_{\mu\nu\rho\sigma}(x)$ 与 $h_{\mu\nu}(x)$ 的关系式。

(c) 证明 $R'_{\mu\nu\rho\sigma}(x')=R_{\mu\nu\rho\sigma}(x)$。

(d) 写出在 $\partial^{\mu}h_{\mu\nu}=0$ 的规范下 $h_{\mu\nu}$ 的运动方程。

解：

(a) 坐标变换下

$$g'_{\mu\nu}(x')=\frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}g_{\rho\sigma}(x)$$

$$\eta'_{\mu\nu}+h'_{\mu\nu}(x')=\left(\delta^{\rho}{}_{\mu}-\frac{\partial\epsilon^{\rho}}{\partial x'^{\mu}}\right)\left(\delta^{\sigma}{}_{\nu}-\frac{\partial\epsilon^{\sigma}}{\partial x'^{\nu}}\right)(\eta_{\rho\sigma}+h_{\rho\sigma})$$

考虑到闵氏度规在这个坐标变换下不变，且 $\frac{\partial}{\partial x'}=\frac{\partial}{\partial x^{\mu}}$ 近似成立，故忽略二阶及以上小量时

$$h'_{\mu\nu}(x')\simeq\delta^{\rho}{}_{\mu}\delta^{\sigma}{}_{\nu}h_{\rho\sigma}(x)-\frac{\partial\epsilon^{\rho}}{\partial x^{\mu}}\eta_{\rho\nu}-\frac{\partial\epsilon^{\sigma}}{\partial x^{\nu}}\eta_{\mu\sigma}=h_{\mu\nu}(x)-\partial_{\mu}\epsilon_{\nu}-\partial_{\nu}\epsilon_{\mu}$$

(b) 用度规表示克里斯多菲符号有（忽略二阶及以上小量）

$$\begin{aligned}\Gamma^{v}{}_{\mu\lambda}&=\frac{1}{2}g^{\nu\sigma}\left(g_{\mu\sigma,\lambda}+g_{\lambda\sigma,\mu}-g_{\mu\lambda,\sigma}\right)\\&\simeq\frac{1}{2}\left(\eta^{\nu\sigma}-h^{\nu\sigma}\right)\left(h_{\mu\sigma,\lambda}+h_{\lambda\sigma,\mu}-h_{\mu\lambda,\sigma}\right)\\&\simeq\frac{1}{2}\eta^{\nu\sigma}\left(h_{\mu\sigma,\lambda}+h_{\lambda\sigma,\mu}-h_{\mu\lambda,\sigma}\right)\end{aligned}$$

所以黎曼曲率写为以下形式（后两项为高阶项略去）

$$\begin{aligned}R^{\nu}{}_{\mu\kappa\lambda}&=\partial_{\kappa}\Gamma^{\nu}{}_{\mu\lambda}-\partial_{\lambda}\Gamma^{\nu}{}_{\mu\kappa}+\Gamma^{\nu}{}_{\sigma\kappa}\Gamma^{\sigma}{}_{\mu\lambda}-\Gamma^{\nu}{}_{\sigma\lambda}\Gamma^{\sigma}{}_{\mu\kappa}\\&\simeq\frac{1}{2}\eta^{\nu\sigma}\left(h_{\mu\sigma,\lambda\kappa}+h_{\lambda\sigma,\mu\kappa}-h_{\mu\lambda,\sigma\kappa}\right)-\frac{1}{2}\eta^{\nu\sigma}\left(h_{\mu\sigma,\kappa\lambda}+h_{\kappa\sigma,\mu\lambda}-h_{\mu\kappa,\sigma\lambda}\right)\\&=\frac{1}{2}\eta^{\nu\sigma}\left(h_{\lambda\sigma,\mu\kappa}+h_{\mu\kappa,\sigma\lambda}-h_{\mu\lambda,\sigma\kappa}-h_{\kappa\sigma,\mu\lambda}\right)\end{aligned}$$

$$\begin{aligned}R_{\nu\mu\kappa\lambda}&=g_{\nu\sigma}R^{\sigma}{}_{\mu\kappa\lambda}=\frac{1}{2}(\eta_{\nu\sigma}+h_{\nu\sigma})\eta^{\sigma\alpha}\left(h_{\lambda\alpha,\mu\kappa}+h_{\mu\kappa,\alpha\lambda}-h_{\mu\lambda,\alpha\kappa}-h_{\kappa\alpha,\mu\lambda}\right)\\&\simeq\frac{1}{2}\left(h_{\lambda\nu,\mu\kappa}+h_{\mu\kappa,\nu\lambda}-h_{\mu\lambda,\nu\kappa}-h_{\kappa\nu,\mu\lambda}\right)\end{aligned}$$

即

$$R_{\mu\nu\rho\sigma}=\frac{1}{2}\left(h_{\mu\sigma,\nu\rho}+h_{\nu\rho,\mu\sigma}-h_{\nu\sigma,\mu\rho}-h_{\mu\rho,\nu\sigma}\right)$$

(c)

$$\begin{aligned}
R'_{\mu\nu\rho\sigma}(x') &= \frac{1}{2}\left(h'_{\mu\sigma,\nu\rho}+h'_{\nu\rho,\mu\sigma}-h'_{\nu\sigma,\mu\rho}-h'_{\mu\rho,\nu\sigma}\right)\\
&= \frac{1}{2}\left(h_{\mu\sigma,\nu\rho}+h_{\nu\rho,\mu\sigma}-h_{\nu\sigma,\mu\rho}-h_{\mu\rho,\nu\sigma}\right)\\
&+ \frac{1}{2}\left(-\epsilon_{\sigma,\mu\nu\rho}-\epsilon_{\mu,\sigma\nu\rho}-\epsilon_{\rho,\nu\mu\sigma}-\epsilon_{\nu,\rho\mu\sigma}+\epsilon_{\sigma,\nu\mu\rho}+\epsilon_{\nu,\sigma\mu\rho}+\epsilon_{\rho,\mu\nu\sigma}+\epsilon_{\mu,\rho\nu\sigma}\right)\\
&= \frac{1}{2}\left(h_{\mu\sigma,\nu\rho}+h_{\nu\rho,\mu\sigma}-h_{\nu\sigma,\mu\rho}-h_{\mu\rho,\nu\sigma}\right)\\
&= R_{\mu\nu\rho\sigma}(x)
\end{aligned}$$

(d) 里奇张量为

$$R_{\beta\nu}=\eta^{\alpha\mu}R_{\alpha\beta\mu\nu}=\frac{1}{2}\left(h^{\mu}{}_{\nu,\beta\mu}+h_{\beta}{}^{\alpha}{}_{,\alpha\nu}-h_{\alpha}{}^{\alpha}{}_{,\beta\nu}-h_{\beta\nu,\alpha}{}^{\alpha}\right)$$

曲率标量就是

$$R=\eta^{\beta\nu}R_{\beta\nu}=\frac{1}{2}\left(h^{\mu\beta}{}_{,\beta\mu}+h^{\nu\alpha}{}_{,\alpha\nu}-h_{\alpha}{}^{\alpha}{}_{,\beta}{}^{\beta}-h_{\beta}{}^{\beta}{}_{,\alpha}{}^{\alpha}\right)=h^{\mu\beta}{}_{,\beta\mu}-h_{\alpha}{}^{\alpha}{}_{,\beta}{}^{\beta}$$

约定 $h=\eta^{\mu\nu}h_{\mu\nu}$，给出爱因斯坦张量

$$\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\\
&= -\frac{1}{2}\left(\Box h_{\mu\nu}-\eta_{\mu\rho}h^{\rho\lambda}{}_{,\lambda\nu}-\eta_{\nu\rho}h^{\rho\lambda}{}_{,\lambda\mu}+h_{,\mu\nu}+\eta_{\mu\nu}h^{\alpha\beta}{}_{,\beta\alpha}-\eta_{\mu\nu}\Box h\right)
\end{aligned}$$

取定规范为 $h^{\mu\nu}{}_{,\nu}=0$ 时，线性爱因斯坦方程即度规扰动 $h_{\mu\nu}$ 的运动方程为

$$\Box h_{\mu\nu}+h_{,\mu\nu}-\eta_{\mu\nu}\Box h=-16\pi GT_{\mu\nu}$$

习题 2

$R_g=2GM$ 称为质量为 M 的黑洞的引力半径。若在 $r_o=10R_g$ 处的观察者收到发自静止于 $r_s=5R_g$ 处的光源发出的光，试求接收到的光的频率与发出时频率的比值。（r 为径向坐标）

解：

根据黄超光（3.6.6）式，接受到的光频率 ν_o 与光源发出的光频率 ν_s 的比值为

$$\frac{\nu_0}{\nu_s}=\frac{(1-2GM/r_s)^{1/2}}{(1-2GM/r_0)^{1/2}}=\frac{(1-2GM/10GM)^{1/2}}{(1-2GM/20GM)^{1/2}}=\frac{2\sqrt{2}}{3}\approx 0.94$$